Introduction to QPA

Part 4

Øystein Skartsæterhagen   Øyvind Solberg

Department of Mathematical Sciences
Norwegian University of Science and Technology

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Outline

1. Examples of computations
2. Tilting modules

Ω-periodic modules

Λ – finite dimensional algebra

Recall: A module M is Ω-*periodic* if $\Omega^n_\Lambda(M) \simeq M$ for some positive integer n.

In some situations the period can indicate the degree of generators in the Hochschild cohomology ring, $\bigoplus_{i \geqslant 0} \operatorname{Ext}^i_{\Lambda^{\mathrm{env}}}(\Lambda, \Lambda)$.

Periodic algebras

Λ – finite dimensional k-algebra

$\Lambda^{\mathrm{env}} = \Lambda^{\mathrm{op}} \otimes_k \Lambda$ – enveloping algebra

Recall: Λ is a *periodic algebra* if Λ is a Ω-periodic module, that is $\Omega^n_{\Lambda^{\mathrm{env}}}(\Lambda) \simeq \Lambda$ as Λ^{env}-modules.

Facts:

- Λ is a selfinjective algebra.
- All modules are Ω-periodic.
- The Hochschild cohomology modulo nilpotent elements is isomorphic to $k[x]$, where the degree of x is the period.

Finding quivers

Recall: $\Lambda = kQ/I$ with I admissible is a basic algebra, that is,

$$\Lambda = \oplus_{i=1}^{t} P_i$$

with P_i indecomposable, then $P_i \not\simeq P_j$ for $i \neq j$.

Facts:

- Λ – finite dimensional algebra.
- $\operatorname{rad} \Lambda = \langle \text{arrows} \rangle / I$.
- $\Lambda / \operatorname{rad} \Lambda \simeq \frac{kQ/I}{\langle \text{arrows} \rangle / I} \simeq \text{linear span of vertices}$.
- $$\begin{aligned} \operatorname{rad} \Lambda / \operatorname{rad}^2 \Lambda &\simeq \frac{\langle \text{arrows} \rangle}{I} \Big/ \frac{\langle \text{arrows} \rangle^2}{I} \\ &\simeq \frac{\langle \text{arrows} \rangle}{\langle \text{arrows} \rangle^2} \\ &\simeq \text{linear span of arrows} \end{aligned}$$

Finding quivers

Λ – finite dimensional algebra with $\Lambda/\operatorname{rad}\Lambda \simeq k^n$ for some n.

Algorithm:

1. Lift a complete set of orthogonal idempotents from $\Lambda/\operatorname{rad}\Lambda$ to a complete set of orthogonal idempotents in Λ, say $\{e_i\}_{i=1}^n$ – the vertices.
2. Compute $e_i \operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda e_j$, find a basis and lift back to $e_i \operatorname{rad}\Lambda e_j$ – the arrows from vertex i to vertex j.
3. Construct a quiver Q from this and a homomorphism $\varphi\colon kQ \to \Lambda$.
4. Find the kernel of φ.

Trivial extensions

Λ – finite dimensional algebra

$T(\Lambda) = \Lambda \oplus D(\Lambda)$ – *trivial extension*,

$$(\lambda, f) \cdot (\lambda', f') = (\lambda\lambda', \lambda f' + f\lambda)$$

- $\operatorname{rad} T(\Lambda) = \operatorname{rad} \Lambda \oplus D(\Lambda)$.
- $\operatorname{rad}^2 T(\Lambda) = \operatorname{rad}^2 \Lambda \oplus D(\Lambda) \operatorname{rad} \Lambda + \operatorname{rad} \Lambda D(\Lambda)$.
- $\frac{\operatorname{rad} T(\Lambda)}{\operatorname{rad}^2 T(\Lambda)} \simeq \frac{\operatorname{rad} \Lambda}{\operatorname{rad}^2 \Lambda} \oplus \frac{D(\Lambda)}{D(\Lambda) \operatorname{rad} \Lambda + \operatorname{rad} \Lambda D(\Lambda)}$
- $T(\Lambda)$ is a symmetric algebra, $\Gamma \simeq D(\Gamma)$ as bimodules.

AR-theory

Recall that a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *almost split exact* if it is not split exact and

(i) for any not splittable epimorphism $t\colon X \to C$ there is a homomorphism $t'\colon X \to B$ such that $gt' = t$,
(ii) for any not splittable monomorphism $s\colon A \to Y$ there is a homomorphism $s'\colon A \to Y$ such that $s'f = s$.

AR-theory

Facts:

- C and A are indecomposable modules.
- $A \simeq D\operatorname{Tr} C$ and $C \simeq \operatorname{Tr} D(A)$.
- For any indecomposable non-projective module C and for any indecomposable non-injective module A, there is an almost split sequence ending in C and starting in A.
- An almost split sequence is a generator of the socle of $\operatorname{Ext}^1_\Lambda(C, D\operatorname{Tr}(C))$ as an $\operatorname{End}_\Lambda(C)$-module.

APR-tilting

$\Lambda = kQ$ – hereditary, Q no oriented cycle and connected.

S simple projetive module (and not injective)
Λ is a classical tilting module T:

- $\mathrm{pd}_\Lambda T \geqslant 1$,
- $\mathrm{Ext}^1_\Lambda(T, T) = (0)$,
- the number of indecomposable non-isomorphic summands in T is equal to the number of isomorphism class of simple modules of Λ.

$\Lambda = P \oplus S \longrightarrow T = P \oplus \mathrm{Tr}\, D(S)$ – APR-tilting